5

Fourier Series in Diffusion and Wave Phenomena

One of the main fields of application of Fourier series is in finding the solution of processes governed by linear partial differential equations where the space derivative is the Laplacian. In such processes, it is the *local curvature* of the disturbance which is subject to the time development as determined by the time derivatives. If the latter is a first-order derivative, we have the diffusion equation [Eq. (5.1)], where the *rate* of change in temperature is proportional to its local curvature. In the wave equation [Eq. (5.15)], it is the *acceleration*, the second time derivative, which responds linearly to the disturbance curvature. If the boundary conditions are *periodic* with some period $2L$, Fourier series will provide an expansion of the solution in terms of a basis of Laplacian eigenfunctions with exactly these periodicity conditions.

The diffusion equation is analyzed in Section 5.1, and in Section 5.2 the wave equation is presented. The boundary conditions proposed in the latter are those of a fixed-end string rather than those of a vibrating ring, say. This is done partly because of the general interest of *constrained* elastic media and partly for the opportunity it provides to illustrate the use of the Fourier sine series. In both cases we present several approaches: (a) the Green's function treatment, (b) normal modes, (c) hyperdifferential time-evolution operators, and (d) for the wave equation, traveling waves. In Section 5.3 we apply Fourier series to describe a mechanical lattice composed of an infinity of masses and springs.

5.1. Heat Diffusion in a Ring

In this section we derive the diffusion equation from physical considerations about heat conduction. This partial differential equation is easily solved by Fourier series, the theta function of Section 4.4 being the Green's function for the system. Operator methods, introduced later, will be seen to abbreviate the derivation.

5.1.1. The Heat Equation

A homogeneous conducting medium whose temperature $f(\mathbf{x}, t)$ is a function of the point $\mathbf{x}$ at time t will satisfy the *heat equation*

$$\frac{\partial}{\partial t} f(\mathbf{x}, t) = a^2 \nabla^2 f(\mathbf{x}, t), \qquad a^2 = \kappa/\mu s, \tag{5.1}$$

where a is the diffusion constant given in terms of the conductivity κ, mass density μ, and the specific heat s of the medium. Equation (5.1) states that the rate of change of temperature with time at a point is proportional to the local curvature of the function in the direction of concavity ("nature hates vacua"). The constant a^2 describes the time scale of the diffusion process.

We shall sketch how Eq. (5.1) arises in a one-dimensional ("thin rod") medium. See Fig. 5.1. The *heat flux* $\Phi(x, t)$ across a point x (in calories per unit time) is observed to be proportional to the *temperature gradient* at x (in °K per unit length), i.e., $\Phi(x, t) = -\kappa\, \partial f(x, t)/\partial x$, the proportionality constant κ being the conductivity of the medium and the minus sign indicating that heat flows from warmer to colder regions. The *net flux* of heat into the segment extending from x to $x + \Delta x$ is $\Phi_{\text{net}}(x, t) = \Phi(x, t) - \Phi(x + \Delta x, t)$ and results in a change of temperature. The number of calories needed to raise the rod element mean temperature by 1°K is given by the specific heat s of the material times the linear mass density μ times the length Δx. Thus $\Phi_{\text{net}}(x, t) = \mu s\, \Delta x\, \partial f(x, t)/\partial t$. Equating the expressions involving the temperature, dividing by Δx, and letting $\Delta x \to 0$, one obtains Eq. (5.1) in one dimension. The basic arguments outlined here can be repeated for heat diffusion in two, three, or more dimensions.

The differential equation (of *parabolic* type) in Eq. (5.1) still has to be complemented by boundary conditions in time and space in a manner which

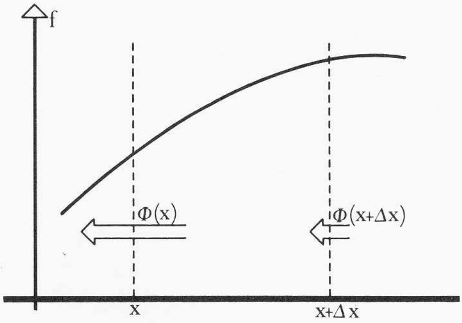

Fig. 5.1. Temperature and heat flux in a thin rod.

will be brought out below. In what follows, we shall absorb the constant a^2 into rescaling the time units. In the resulting formulas, the diffusion constant a can be regained by replacing t by a^2t.

Equation (5.1) for the temperature $f(\mathbf{x}, t)$ can be multiplied on both sides by the *heat capacity* C (in calories per °K per unit volume), giving an identical equation for $\rho(\mathbf{x}, t) := Cf(\mathbf{x}, t)$; the amount of heat per unit volume Equation (5.1) then describes a compressible but nonevanescent fluid which can represent such diverse processes as the interpenetration of one liquid by another or the diffusion of neutrons through matter.

5.1.2. Solution by Fourier Series

The boundary conditions we use here to illustrate the use of Fourier series describe a conducting *ring* of unit radius, where x represents the arc length. A ring with arbitrary radius does not introduce any novel features: it can be easily treated using the form (4.132) for the series. The boundary conditions in space are thus $f(x, t) = f(x + 2\pi, t)$, and the temperature function can be taken to represent a vector $\mathbf{f}(t)$ in the function space described in Chapter 4. Equation (5.1) thus becomes the vector equation (with rescaled time)

$$\frac{d}{dt}\mathbf{f}(t) = \nabla^2\mathbf{f}(t). \tag{5.2}$$

In the $\boldsymbol{\varphi}$-basis, (5.2) implies the equality of the corresponding column-vector coefficients. Those on the left-hand side are $\dot{f}_n(t) := df_n(t)/dt$, while those on the right can be found from (4.51). Hence (5.1) plus the periodic boundary conditions are equivalent to the set of equations

$$\dot{f}_n(t) = -n^2 f_n(t), \qquad n \in \mathscr{Z}. \tag{5.3}$$

[The process of finding (5.3) from (5.1) is analogous to the uncoupling of the lattice equations of motion in Chapter 2. From the second-order *partial* differential equation (5.1) we thus find an (infinite) set of first-order *ordinary* differential equations. The x and t derivatives are now uncoupled. The interaction operator is the Laplacian ∇^2, in correspondence with the second-difference operator which appeared in Chapter 2.]

The general solution of (5.3) is of the type $c_n \exp(-n^2t)$, with arbitrary constants c_n which are fixed when the initial conditions in time are specified. For $t = t_0$, let the temperature be $\mathbf{f}(t_0)$ with Fourier components $f_n(t_0)$. The constants c_n can then be uniquely evaluated in terms of the initial condition yielding

$$f_n(t) = f_n(t_0) \exp[-n^2(t - t_0)], \qquad n \in \mathscr{Z}, \tag{5.4}$$

as the general solution of (5.3). The original temperature function $f(x, t)$ can finally be regained as the Fourier synthesis of (5.4), i.e., by Eq. (4.32a),

$$f(x, t) = (2\pi)^{-1/2} \sum_{n \in \mathscr{Z}} f_n(t) \exp(inx). \tag{5.5}$$

5.1.3. The Green's Function and Fundamental Solutions

We note that the Fourier coefficients (5.4) of $f(x, t)$ are the product of the Fourier coefficients of $f(x, t_0)$ times $\exp[-n^2(t - t_0)]$, which are the Fourier coefficients of the theta function $\theta(x, t - t_0)$ in Eq. (4.64), times $(2\pi)^{1/2}$. The temperature function (5.5) will thus be the *convolution* of the two, i.e.,

$$\begin{aligned} f(x, t) &= [\theta(\cdot, t - t_0) * f(\cdot, t_0)](x) \\ &= \int_{-\pi}^{\pi} dx' \theta(x - x', t - t_0) f(x', t_0). \end{aligned} \tag{5.6}$$

This expression has a very transparent physical meaning. To bring this out, consider the special (unphysical) case where the initial conditions are $f(x', t_0) = \delta(x' - x_0)$, i.e., an infinitely hot spot at x_0. The temperature thereafter is then given by (5.6) as the *fundamental solution*

$$f(x, t) = \theta(x - x_0, t - t_0), \qquad t \geqslant t_0, \tag{5.7}$$

which is a theta function centered at x_0. See Fig. 4.13.

If the initial temperature distribution were a finite collection of hot points at x_i, that is, $\sum_i f_i \delta(x - x_i)$, the resulting solution would be a sum of θ's centered at x_i with coefficients f_i. An arbitrary initial condition $f(x, t_0)$ can be seen as a sum—à la Riemann, gone to the limit—of δ's distributed over $x' \in (-\pi, \pi]$ with coefficients $f(x', t_0)\, dx'$. The resulting temperature distribution is then (5.6). The theta function is thus the *Green's function* for diffusive processes; it appears as an integral kernel in (5.6) and relates the initial condition at (x', t_0) and its effect at (x, t). It has the properties:

(a) It is an *even* function of space: $\theta(x, t) = \theta(-x, t)$, which means that, preserving their time ordering, the points of cause x' and the points of effect x can be *exchanged*. This is the principle of *reciprocity*.

(b) The effect of x' on x depends only on their *relative* separation $x' - x$, as can be seen in the corresponding functional dependence of the Green's function: the system is *translationally invariant*.

(c) The system is *invariant under inversions* since the Green's function depends only on the absolute value $|x' - x|$. (Compare with Section 2.2.)

The theta function $\theta(x, t - t_0)$ is infinitely differentiable in x and in $t > t_0$ as its Fourier series shows. Since the solution $f(x, t)$ is a convolution of the initial condition with $\theta(x, t - t_0)$, it follows that $f(x, t)$ itself will also be infinitely differentiable in the half-plane (x, t), $t > t_0$.

5.1.4. The Time-Evolution Operator

From the above discussion it follows that the integral kernel given by the Green's function $\theta(x, t)$ acts as a linear operator,

$$\mathbf{f}(t) = \mathbb{G}(t - t_0)\mathbf{f}(t_0), \tag{5.8}$$

mapping the space of generalized functions which are the initial conditions $\mathbf{f}(t_0)$ of the system on the space of infinitely differentiable functions for $t > t_0$.

Since any linear combination of solutions of (5.1) is also a solution to this equation, *the set of all solutions of the diffusion equation constitutes a linear vector space.*

Further properties of the Green's function are that *total heat is preserved* and that the set of Green's functions for all $t > t_0$ constitutes a *semigroup of integral kernels.* This we leave to the reader to verify in Exercises 5.1 and 5.2.

Exercise 5.1. Show that the *total heat* of the system

$$Q := \int_{-\pi}^{\pi} dx f(x, t) \tag{5.9}$$

is a constant, independent of time. This can be proven (a) by substitution of (5.6) into (5.9), exchange of integrals, and the property (4.68) of the theta function, or (b) by calculating the time derivative of (5.9), using the governing equation (5.1) and showing that the evaluated integral is zero due to the periodic boundary conditions in x. Another proof is suggested in Exercise 5.5.

Exercise 5.2. Let the temperature function at time t be due to initial conditions at t_1 and these in turn a consequence of an earlier t_0 temperature distribution. Show that time evolution is a *transitive* process in the sense that

$$\begin{aligned} f(\cdot, t) &= \theta(\cdot, t - t_1) * f(\cdot, t_1) = \theta(\cdot, t - t_1) * \theta(\cdot, t_1 - t_0) * f(\cdot, t_0) \\ &= \theta(\cdot, t - t_0) * f(\cdot, t_0), \end{aligned} \tag{5.10}$$

which is satisfied since

$$\int_{-\pi}^{\pi} dx' \theta(x - x', t_1)\theta(x' - x'', t_2) = \theta(x - x'', t_1 + t_2). \tag{5.11}$$

Equation (5.11) can be proven either directly or by the product of the Fourier coefficients of the θ's in convolution. This associates to every time $t \geqslant 0$ an integral kernel with (a) the composition law (5.11), (b) identity given by the Dirac δ, and (c) associativity. For negative time t the θ-function series is strongly divergent, so the general inverse for the set of integral kernels does not exist. We have thus a *semigroup* of time-evolution operators.

Exercise 5.3. Can a temperature distribution of the form of a rectangle or a triangle function be regressed in time at all? Find a condition so that a temperature distribution allows time regression to $-\tau$. Can any temperature distribution be regressed in time indefinitely? Work in the Fourier basis only.

Exercise 5.4. As Eq. (5.1) manifestly allows, search for its *separable* solutions $f_n(x, t) = X_n(x)T_n(t)$, n specifying the separation constant. By introducing this form into (5.1) and recalling the space boundary conditions, the solutions found will be of the form $\exp(-n^2t + inx)$ for $n \in \mathscr{Z}$. These are the "normal modes" for heat diffusion in the ring. The most general solution will be a sum over n of these solutions with coefficients determined from the initial conditions. Show that one regains the form (5.6) with the series development of the theta function.

5.1.5. Hyperdifferential Form of the Evolution Operator

The solutions of the diffusion equation lend themselves to a general presentation by *hyperdifferential* operators. One can formally expand the solution of (5.1) using the Taylor series in t around t_0 as

$$f(x, t) = \sum_{n=0}^{\infty} \frac{(t - t_0)^n}{n!} \frac{\partial^n}{\partial t'^n} f(x, t')|_{t'=t} =: \exp\left[(t - t_0)\frac{\partial}{\partial t'}\right] f(x, t')|_{t'=t}. \tag{5.12}$$

Now, on the space of solutions of (5.1), the operator $\partial/\partial t$ is equivalent to ∇^2, and hence (5.12) can be expressed as

$$f(x, t) = \exp[(t - t_0)\nabla^2] f(x, t_0), \tag{5.13}$$

which should then be equivalent to the time evolution (5.6) in terms of an integral kernel. [Compare with Eq. (2.38b).] It would appear that (5.13) can hold only when the initial temperature distribution is infinitely differentiable. Actually, (5.13) holds *weakly* for any generalized function $f(x, t_0)$ as can be seen when $f(x, t_0) = \delta(x - x_0)$ so that $f(x, t)$ is the fundamental solution (5.7). The weak equality between the integral convolution (5.6) and the *hyperdifferential* operator in (5.13) was established in (4.100).

We can state quite generally that *the exponentiation of a second-order differential operator is weakly equivalent to the action of an integral kernel*, both representing here the time-evolution operator $\mathbb{G}(t - t_0)$ in Eq. (5.8). In Eq. (4.129) we characterized operators represented by diagonal matrices in the $\boldsymbol{\varphi}$-basis as *convolution* operators. Since any powers or sums thereof are diagonal in this basis, $\mathbb{G}(\tau) = \exp(\tau\nabla^2)$ is clearly such an operator.

Exercise 5.5. Prove total heat conservation using the hyperdifferential form of the solution. Note that (5.9) can be written as $Q = (\mathbf{1}, \mathbf{f}(\cdot, t))$, where $\mathbf{1}$ is the unit constant function in $(-\pi, \pi]$. The Parseval identity then allows us to write Q as an inner product in the $\boldsymbol{\varphi}$-basis, which is manifestly time independent.

Exercise 5.6. Prove the semigroup property (5.11) from the corresponding time-evolution operator product

$$\mathbb{G}(t_1)\mathbb{G}(t_2) = \mathbb{G}(t_1 + t_2), \qquad \mathbb{G}(0) = \mathbb{1}, \tag{5.14}$$

which in turn is an immediate consequence of the hyperdifferential form (5.13).

Lest the solution of the diffusion equation appear trivial, let us remark that the greater practical difficulties in solving Eq. (5.1) appear when realistic boundary conditions are imposed as curves in the (x, t)-plane and when *sources* of heat or fluid are present. The latter case will be taken up in Part III in studying applications of the Fourier and Laplace transforms. The study of *some* boundary conditions will be taken up in the context of separating coordinates for the diffusion equation as an application of canonical transforms in Chapter 10. Meanwhile, two simple boundary conditions which can be reduced to the periodic case are suggested in Exercises 5.7 and 5.8.

Exercise 5.7. Assume one has a conducting rod extending between two "*cold walls*" at $x = 0$ and $x = \pi$ which maintain the conditions $f(0, t) = 0 = f(\pi, t)$ for all t. Since $\exp(t\nabla^2)$ commutes with $\mathbb{I}_0$ [see Eqs. (4.121)], the "*method of images*" is applicable. It consists of choosing a rod to extend between $x = -\pi$ and π, the segment $(-\pi, 0)$ being the negative mirror image of the temperature function in $(0, \pi)$, i.e., $f(-x, t_0) = -f(x, t_0)$. This relation is preserved for all t. The description can be made using the *sine* Fourier series, Eqs. (4.134).

Exercise 5.8. Assume now that the conducting rod has *insulated* ends at $x = 0$ and π. As there the heat flux is zero, $\partial f(x, t)/\partial x|_{x=0,\pi} = 0$ are the space boundary conditions. The "method of images" with functions symmetric under inversions will use the *cosine* Fourier series (4.135). The reader may find it worthwhile before solving Exercises 5.7 and 5.8 to browse through Section 5.2 where the method of images is used for the wave equation with similar boundary conditions.

5.2. The Vibrating String

Fourier series are well suited for the description of wave phenomena in elastic media with Cartesian boundaries. The disturbance or characteristic $f(\mathbf{x}, t)$ of the medium we want to analyze will be governed by the *wave equation*

$$c^{-2}\frac{\partial^2}{\partial t^2}f(\mathbf{x}, t) = \nabla^2 f(\mathbf{x}, t), \tag{5.15}$$

where c is a constant which will turn out to be the propagation velocity. Equation (5.15) has to be complemented by *boundary conditions* in space and time, typically

$$f(\mathbf{x}, t) = 0 \text{ for } \mathbf{x} \in B, f(\mathbf{x}, t_0) = u(\mathbf{x}), \dot{f}(\mathbf{x}, t_0) = v(\mathbf{x}), \tag{5.16}$$

where B is a fixed boundary enclosing a finite region in $\mathbf{x}$-space. Here we shall concentrate on finding the solutions of (5.15)–(5.16) describing a finite string with fixed ends. Various other boundary conditions and regions will be presented in Chapter 6.

5.2.1. The Wave Equation

Consider a thin string of linear mass density μ stretched with tension τ between two points and allowed to undergo longitudinal or small transverse vibrations. Let $f(x, t)$ be the elongation from equilibrium of the point x of the string at time t. Isolating the string element which extends from x to $x + \Delta x$ (Fig. 5.2, where the elongation is represented as transversal), we see that it is subject only to a net restitution force in the direction of the displacement. At x the force is $-\tau\, \partial f(x, t)/\partial x$, while at $x + \Delta x$ it is $\tau\, \partial f(x', t)/\partial x'|_{x'=x+\Delta x}$. The net force is the sum of these two and will produce an acceleration $-\partial^2 f(x, t)/\partial t^2$ on the mass $\mu\Delta x$ of the element. Using Newton's laws, dividing by Δx, and letting $\Delta x \to 0$, we obtain the wave equation (5.15) with $c^2 := \tau/\mu$. Dimensional analysis shows that c has units of velocity. We could absorb this constant into a redefinition of time units, but we prefer here to leave it appearing explicitly in the ensuing developments.

The boundary conditions which describe a string of length L with fixed endpoints are

$$f(0, t) = 0, \qquad f(L, t) = 0 \qquad \text{for all } t. \tag{5.17}$$

5.2.2. Eigenfunctions of the Laplacian

Since this chapter deals with applications of Fourier series, we can expect that the use of this series will solve the problem posed by (5.15)–(5.17). In fact it does: if we follow the approach used in Section 5.1, we will find that the partial differential wave equation is reduced to a set of ordinary (second-order) differential equations. We would like to present here a line

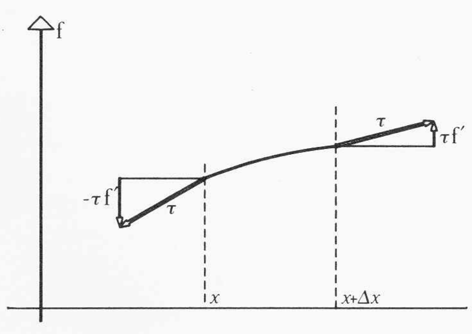

Fig. 5.2. Elongation and tension of an elastic string element.

of reasoning which is somewhat different and which, although leading to the same results, will generalize more easily to the solution of the systems posed in Chapter 6, where the boundary conditions are those of a two-dimensional rectangular, circular, annular, or sectorial membrane.

It is readily verified that in the Hilbert space of functions $\mathscr{L}_0{}^2(0, L)$ with the inner product (4.134c) on $(0, L)$ and boundary conditions (5.17), the operator ∇^2 is hermitian. For twice-differentiable $\mathbf{g}$, $\mathbf{f}$, integrating by parts,

$$\begin{aligned}(\mathbf{g}, \nabla^2\mathbf{f})_L^\circ &= \int_0^L dx g(x)^* \frac{\partial^2 f(x)}{\partial x^2} \\ &= g(x)^* \frac{\partial f(x)}{\partial x}\bigg|_0^L - \int_0^L dx \left[\frac{\partial g(x)}{\partial x}\right]^* \frac{\partial f(x)}{\partial x} \\ &= \left\{g(x)^* \frac{\partial f(x)}{\partial x} - \left[\frac{\partial g(x)}{\partial x}\right]^* f(x)\right\}\bigg|_0^L + \int_0^L dx \left[\frac{\partial^2 g(x)}{\partial x^2}\right]^* f(x) \\ &= 0 + (\nabla^2\mathbf{g}, \mathbf{f})_L^\circ. \end{aligned} \tag{5.18}$$

Moreover, ∇^2 can be shown to be *self-adjoint*. The set of all its eigenvectors

$$\nabla^2 f_\lambda(x) = \lambda f_\lambda(x) \tag{5.19}$$

will constitute a complete orthogonal basis for that space (Section 4.7).

The solutions of the differential equation (5.19) have the general form

$$a \sin[(-\lambda)^{1/2}x] + b \cos[(-\lambda)^{1/2}x], \qquad a, b, \lambda \in \mathscr{C}. \tag{5.20}$$

If we impose the boundary conditions (5.17) at $x = 0$, we obtain the restriction $b = 0$, while the condition at $x = L$ requires $(-\lambda)^{1/2}L \equiv 0 \bmod \pi$, i.e., $(-\lambda)^{1/2}L = n\pi$, $n \in \mathscr{Z}$, or $\lambda = -(n\pi/L)^2$. The eigenfunctions of ∇^2 are thus $\sin(n\pi x/L)$ in $\mathscr{L}_0{}^2(0, L)$, and we can use n to label the eigenfunctions. The values $+|n|$ and $-|n|$ yield the same function, while for $n = 0$ we obtain the zero function. Hence we let $n = 1, 2, 3, \ldots$. The constant a in (5.20) may depend on n and t, so we let $a = a_n(t)$.

We can thus expand any function $f(x, t) \in \mathscr{L}_0{}^2(0, L)$ satisfying (5.15) in terms of eigenfunctions of ∇^2 with these boundary conditions as

$$f(x, t) = (2/L)^{1/2} \sum_{n \in \mathscr{Z}^+} a_n(t) \sin(n\pi x/L), \tag{5.21a}$$

introducing the constant $(2/L)^{1/2}$ in order to match exactly Eqs. (4.134). These allow us to solve for the $a_n(t)$:

$$a_n(t) = (2/L)^{1/2} \int_0^L dx f(x, t) \sin(n\pi x/L) = f_n{}^\circ(t). \tag{5.21b}$$

Equations (5.21) do not yet describe solutions of the wave equation (5.15); they are only an expansion tailored for this equation plus boundary conditions. Now, upon requiring that (5.21a) be a solution to (5.15), we find a set of uncoupled ordinary differential equations for $a_n(t)$ to satisfy, viz.,

$$\begin{aligned} 0 &= \left(c^{-2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) f(x, t) \\ &= (2/L)^{1/2} \sum_{n\in\mathscr{Z}^+} \left(c^{-2}\frac{\partial^2}{\partial t^2} + \lambda_n\right) f_n^{\circ}(t) \sin(n\pi x/L). \end{aligned} \tag{5.22}$$

Linear independence of the eigenvectors of ∇^2 now implies that each of the coefficients of the series (5.22) is zero. The $f_n^{\circ}(t)$ are thus determined up to two arbitrary constants, which we write, introducing for later use an "initial time" t_0, as

$$f_n^{\circ}(t) = b_n \sin[\omega_n(t - t_0)] + c_n \cos[\omega_n(t - t_0)], \qquad b_n, c_n \in \mathscr{C}, \tag{5.23a}$$

$$\omega_n := n\pi c/L, \qquad n \in \mathscr{Z}^+. \tag{5.23b}$$

5.2.3. Initial Conditions and Green's Function

The series (5.21a) with coefficients (5.23) is the most general solution of the problem. It remains now to fix the constants b_n and c_n in terms of boundary conditions in time. Any pair of initial conditions on $f(x, t)$ or its time derivatives for fixed t will be suitable. The most common pair is the initial elongation $f(x, t_0)$ and velocity $\dot{f}(x, t_0)$ for t_0. By (5.21b) and (5.23) this determines the b_n and c_n in terms of the sine Fourier coefficients of the two initial conditions. We find

$$b_n = \dot{f}_n^{\circ}(t_0)/\omega_n, \qquad c_n = f_n^{\circ}(t_0), \tag{5.24}$$

so that upon substitution of (5.24) into (5.23) and (5.23) into (5.21a), the solution can be expressed as

$$\begin{aligned} f(x, t) &= (2/L)^{1/2} \sum_{n\in\mathscr{Z}^+} \omega_n^{-1} \sin[\omega_n(t - t_0)] \sin(n\pi x/L) \dot{f}_n^{\circ}(t_0) \\ &\quad + (2/L)^{1/2} \sum_{n\in\mathscr{Z}^+} \cos[\omega_n(t - t_0)] \sin(n\pi x/L) f_n^{\circ}(t_0) \\ &=: \sum_{n\in\mathscr{Z}^+} G_n^{\circ}(x, t - t_0) \dot{f}_n^{\circ}(t_0) + \sum_{n\in\mathscr{Z}^+} \dot{G}_n^{\circ}(x, t - t_0) f_n^{\circ}(t_0) \\ &= (\mathbb{T}_{-x}\mathbf{G}(t - t_0), \dot{\mathbf{f}}(t_0))_L^{\circ} + (\mathbb{T}_{-x}\dot{\mathbf{G}}(t - t_0), \mathbf{f}(t_0))_L^{\circ} \\ &= \int_0^L dx' G(x - x', t - t_0) \dot{f}(x', t_0) + \int_0^L dx' \dot{G}(x - x', t - t_0) f(x', t_0). \end{aligned} \tag{5.25}$$

The last two equalities deserve comment. The second term in (5.25) contains two sums, each of which gives rise to an inner product (4.134c). The first one involves $\mathbf{f}(t_0)$ with a vector with sine Fourier components

$$\begin{aligned} G_n{}^{o}(x, t - t_0) &:= (2/L)^{1/2}\omega_n^{-1} \sin[\omega_n(t - t_0)] \sin(n\pi x/L) \\ &= (\mathbb{T}_{-x}\mathbf{R}^{[2c(t-t_0),1/2c]})_n{}^{o} = G_n{}^{o}(x, t - t_0)^*. \end{aligned} \tag{5.26a}$$

We recognize in the expression (5.26a) the sine Fourier coefficients of the rectangle function of width $2c(t - t_0)$, height $1/2c$, and centered at x as obtained in Eq. (4.135). [Recall the remark about the negative "phantom" function in $(-L, 0)$.] The second sum in (5.25) involves the time derivative of (5.26a),

$$\begin{aligned} \dot{G}_n{}^{o}(t - t_0) &= (2/L)^{1/2} \cos[\omega_n(t - t_0)] \sin(n\pi x/L) \\ &= \tfrac{1}{2}(\mathbb{T}_{x+c(t-t_0)}\boldsymbol{\delta} + \mathbb{T}_{x-c(t-t_0)}\boldsymbol{\delta})_n{}^{o}, \end{aligned} \tag{5.26b}$$

which we recognize as the coefficients of two δ's sitting at $x + ct$ and $x - ct$. The last equality in (5.25) expresses the convolution of the vector $\dot{\mathbf{f}}(t_0)$, represented by the function $\dot{f}(x', t_0)$, initial velocity, and (5.26a), which is the *Green's function* for the system at hand,

$$G(x - x', \tau) = R^{(2c\tau,1/2c)}(x - x'), \tag{5.27a}$$

and that of the initial condition $f(x', t_0)$ and (5.26b),

$$\dot{G}(x - x', \tau) = \tfrac{1}{2}[\delta(x' - (x + c\tau)) + \delta(x' - (x - c\tau))], \tag{5.27b}$$

integrated over x'.

5.2.4. Fundamental Solutions

To bring out the meaning and properties of Green's function we shall consider the *fundamental solutions* below. [Compare these results with those in Section 2.3.] Assume that initially the string starts from *rest* with a δ-*like* "shape" at some point x_0, i.e., $\dot{f}(x', t_0) = 0$, $f(x', t_0) = \delta(x' - x_0)$. The ensuing development of the string shape is then $\dot{G}(x - x_0, t - t_0)$. Equation (5.27b) tells us that the δ-pulse splits into two pulses traveling along $x = x_0 \pm c(t - t_0)$, i.e., they keep their δ-shape at all times and propagate with velocity $\pm c$. Such a pulse is shown in Fig. 5.3(a). Assume next that the string starts from zero elongation, $f(x, t_0) = 0$, but with a δ-pulse in velocity at some x_0, $\dot{f}(x, t_0) = \delta(x - x_0)$, as if impelled by a sharp, localized blow. The string shape will then develop as $G(x - x_0, t - t_0)$, shown in Fig. 5.3(b); it is a rectangle function which broadens with velocity c. The most general solution with initial conditions given by $f(x, t_0)$ and $\dot{f}(x, t_0)$ will be an integral—a generalized linear combination—of these fundamental solutions.

Exercise 5.9. Verify that (5.27b) is the time derivative of (5.27a). You can write $R^{(2ct,1)}(x) = \Theta(ct - x)\Theta(ct + x)$, where Θ is the Heaviside step function [$\Theta(y) = 1$ for $y > 0$, $\Theta(0) = \frac{1}{2}$, $\Theta(y) = 0$ otherwise for $y \in (0,L)$], and use the fact that the derivative of a discontinuous function is a Dirac δ.

We recognize the following properties of the Green's function, which hold for the lattice of Section 2.2 or the diffusive systems in Section 5.1: (a) reciprocity, (b) translational, and (c) inversion invariance. In addition, the system exhibits (d) *causality*: a disturbance at (x_0, t_0) can affect only those points x at future times t which are inside the *cone* $|x - x_0| \leqslant c|t - t_0|$. Both the Green's function and its derivative are zero outside this region.

Exercise 5.10. Let the string elongation at time t depend on conditions at time t_1 and these in turn on still earlier initial conditions at time t_0. Express this transitive property in terms of an integral relation between the Green's function and its derivative for times t, t_1, and t_0. Refer to Exercises 2.13 and 5.2 and ahead to Exercise 5.17.

5.2.5. Traveling Waves and Reflection Phenomena

As Figs. 5.3(a) and (b) suggest, something rather dramatic happens when the disturbance traveling with velocity $\pm c$ hits the endpoints of the string. These are kept fixed, and the pulse undergoes a *reflection*, propagating backwards after the collision. Rather than unearth this phenomenon from the Green's function, we can show rather easily what the mechanism is. For this it is sufficient to note that if $g^{\rightarrow}(y)$ and $g^{\leftarrow}(y)$ are two arbitrary functions,

$$f(x, t) = g^{\rightarrow}(x - ct) + g^{\leftarrow}(x + ct) \tag{5.28}$$

will be a solution of the wave equation (5.15). In fact, the most general solution can be built in this way: a right-moving disturbance plus a left-moving one. The boundary conditions (5.17) impose $g^{\rightarrow}(-ct) = -g^{\leftarrow}(ct)$ and $g^{\rightarrow}(L - ct) = -g^{\leftarrow}(L + ct)$, which can be combined as

$$g^{\rightarrow}(y) = -g^{\leftarrow}(-y) = g^{\rightarrow}(2L + y). \tag{5.29}$$

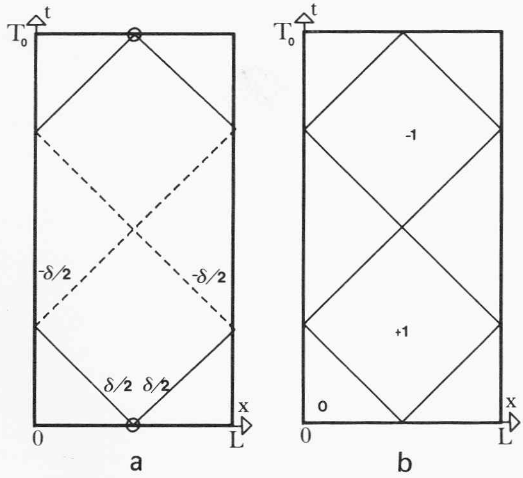

Fig. 5.3. The Green's function and its time derivative for an elastic string of fixed endpoints stretching between 0 and L. (a) $\dot{G}(x - L/2, t)$, (b) $G(x - L/2, t)$. The first consists of traveling Dirac δ's, while the second has values 0, +1, and −1 in the regions shown.

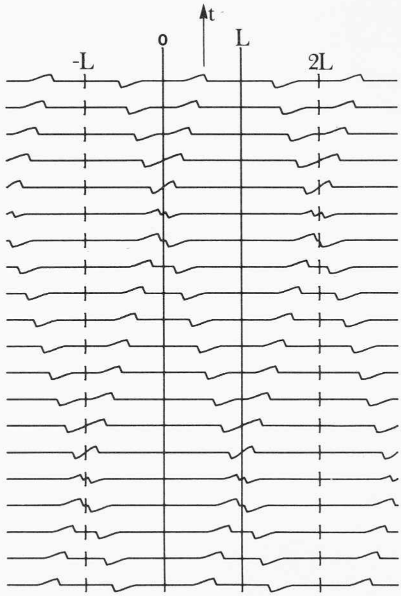

Fig. 5.4. A "lone" traveling pulse in a string undergoing reflection at the endpoints (heavy lines). It is mathematically accompanied by mirror pulses beyond the string ends $0, L$. Reflection thus appears as the entrance of a "mirror" pulse in the "real" string.

This means that the right-moving disturbance must be equal to the negative of the inverted left-moving disturbance, and both must be periodic with period $2L$. Any string movement will be a superposition of two such opposing *traveling waves*. A pulse moving "alone" along the string (Fig. 5.4) is mathematically accompanied by an infinity of companion pulses spaced by $2L$ moving in the same direction and by a second infinity of negative mirror pulses traveling in the opposite direction. When the pulse "hits" the wall, it superimposes with its mirror counterpart. As the pulse proceeds into the mirror region, the mirror pulse becomes real and travels through the string. Reflection has taken place. In Fig. 5.5 we show in detail the reflection process undergone by a moving square pulse.

Exercise 5.11. Show that the above description of companion and mirror images of any string shape is contained in the Green's function formalism from Eq. (5.25) onward. Note that Eq. (5.25) can be rewritten as

$$f(x, t) = (2L)^{-1/2}\Big[\sum_{n\in\mathscr{Z}^+} (f_n{}^{\circ}s_n{}^{+} - \dot{f}_n{}^{\circ}\omega_n^{-1}c_n{}^{+}) - \sum_{n\in\mathscr{Z}^+} (f_n{}^{\circ}s_n{}^{-} - \dot{f}_n{}^{\circ}\omega_n{}^{-}c_n{}^{-})\Big], \tag{5.30a}$$

$$f_n{}^{\circ} = f_n{}^{\circ}(t_0), \qquad \dot{f}_n{}^{\circ} = \dot{f}_n{}^{\circ}(t_0), \tag{5.30b}$$

$$s_n{}^{\pm} := \sin\{n\pi[c(t - t_0) \pm x]/L\}, \qquad c_n{}^{\pm} := \cos\{n\pi[c(t - t_0) \pm x]/L\}. \tag{5.30c}$$

Note that for $t = t_0$ this is the sine and cosine Fourier series for functions of period $2L$ and that as $x \leftrightarrow -x$, $s^+ \leftrightarrow s^-$ and $c^+ \leftrightarrow c^-$; hence $f(-x, t) = -f(x, t)$.

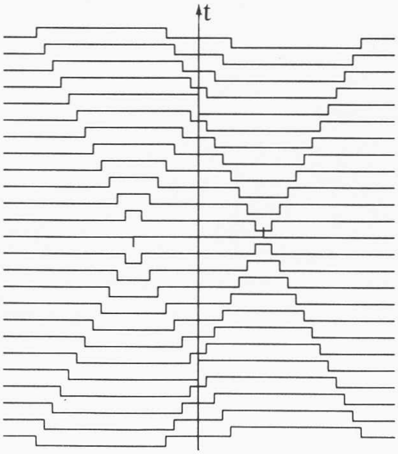

Fig. 5.5. Reflection of a square pulse at a string endpoint. Either half of the figure may represent the "real" string; the other will represent its image.

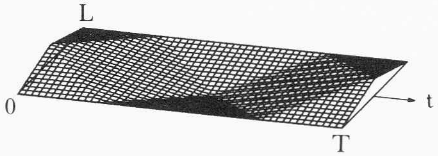

Fig. 5.6. String which starts from rest with a triangle shape.

Exercise 5.12. Explain the development of a string shape which starts from rest in terms of superpositions of right- and left-traveling waves. You can guide yourself with Fig. 5.3(a). Do the same with a string the shape of that in Fig. 5.3(b).

Exercise 5.13. Consider a string which starts from rest with a triangular shape as in Fig. 5.6 as a superposition of right- and left-traveling shapes. Show that the string motion is indeed the one depicted in the figure. At what time does the string recover its initial shape? Did such a "fundamental period" exist for the finite lattice (Chapter 2)?

5.2.6. Normal Modes

The description of the fundamental solutions following Eq. (5.25) was made assuming that the initial displacements and velocities were Dirac δ's. As in Section 2.3, we can now investigate the string motion when the initial conditions are given by the ∇^2 eigenvectors, $(2/L)^{1/2}\sin(n\pi x/L)$, $n \in \mathscr{Z}^+$, in $\mathscr{L}_0^2(0, L)$. The solutions thus obtained are the *normal modes* of the string and can be read from the second member of (5.25), letting the $f_n{}^\circ(t_0)$ and $\dot{f}_n{}^\circ(t_0)$ be different from zero one at a time. Setting $t_0 = 0$ for simplicity, we define

$$\dot{\varphi}_n(x, t) := (2/L)^{1/2}\sin(n\pi x/L)\cos\omega_n t, \qquad \omega_n := n\pi c/L \quad (5.31a)$$

$$\varphi_n(x, t) := (2/L)^{1/2}\sin(n\pi x/L)\omega_n^{-1}\sin\omega_n t, \qquad n \in \mathscr{Z}^+. \quad (5.31b)$$

The most general solution to the string problem is a linear combination of these, as (5.25) can be rewritten in the form

$$f(x, t) = \sum_{n \in \mathscr{Z}^+} f_n{}^\circ \dot{\varphi}_n(x, t) + \sum_{n \in \mathscr{Z}^+} \dot{f}_n{}^\circ \varphi_n(x, t). \tag{5.32}$$

[Note the perfect analogy with (2.48) and (2.50).] A few normal modes (5.31a) have been drawn in Fig. 5.7. Some of their relevant properties are the following: (a) The $\dot{\varphi}_n(x, t)$ represent waveforms which start from rest and maximum elongation, while the $\varphi_n(x, t)$ start from the equilibrium shape with maximum velocity. (b) The nth normal mode presents $n - 1$ *nodes* (i.e., zeros) within the interval $(0, L)$, not counting the endpoints. (c) They oscillate with angular velocities ω_n, Eq. (5.23b), which are *discrete* and directly proportional to n. [In terms of the finite lattice Brillouin diagram of Section 2.3, they are all in the "linear" (low-frequency) region, where $\sin z \sim z$.] (d) The period of oscillation of the nth fundamental mode is

$$T_n = 2\pi/\omega_n = 2L/nc = T_0/n, \qquad T_0 := 2L/c \tag{5.33}$$

and is a submultiple of the *fundamental period* T_0. The original form of *any* string disturbance is thus reproduced after a time T_0, the nth component mode having completed n full oscillations. See again Fig. 5.6. (e) Each normal mode is a sinusoidal string shape modulated by an oscillating function of time: they are the *separated* solutions of the wave equation [i.e., of the form $X_n(x)\tau_n(t)$]. In fact, we would have found precisely these had we set out proposing separated solutions for this equation, the separation constant being proportional to n^2. (f) The odd-n modes are *even* under inversions

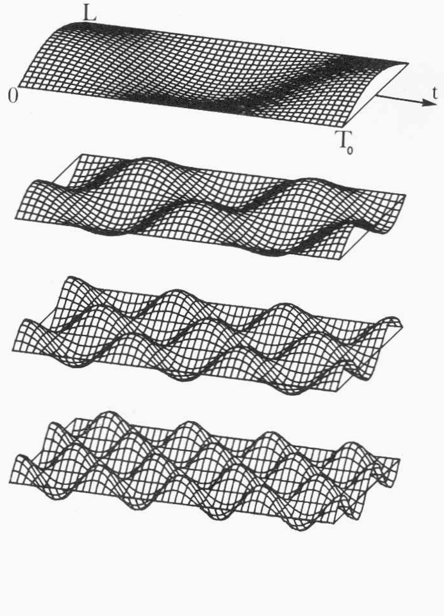

Fig. 5.7. The first four normal modes for a vibrating string that starts from rest [Eq. (5.31a) for $n = 1, 2, 3$, and 4].

through the string midpoint $x = L/2$. Even-n modes are odd. (g) Under time inversion, the $\dot{\varphi}_n(x, t)$ are even, while the $\varphi_n(x, t)$ are odd.

Exercise 5.14. Analyze the string motion in Fig. 5.6 in terms of the constituent normal modes. Show that only the *odd* modes appear. This can be predicted on the basis of the symmetry of the initial conditions with respect to the string midpoint.

Exercise 5.15. Analyze the translation and inversion symmetries of Fig. 5.6: (a) periodicity in time under translations T_0 and $T_0/2$, and in space under translations by L and $2L$; (b) inversions in time through $t = 0$, $T_0/4$,and $T_0/2$, and in space through $x = 0$, $L/4$, and $L/2$.

5.2.7. Two-Component First-Order Differential Form of the Wave Equation

The solutions of the wave equation on the finite string can also be expressed in terms of hyperdifferential operators acting on the initial conditions. This follows a similar treatment of the diffusion problem in Eq. (5.13) but with the introduction of a space of velocity functions $\dot{f}(x, t)$ in addition to the functions $f(x, t)$ which describe the string elongation. [This is analogous to the phase space in Section 2.6.] We consider $\mathbf{f}$ and $\dot{\mathbf{f}}$ as the components of a two-vector $\boldsymbol{\zeta}(x, t)$ so that the wave equation (5.15) appears as a two-component equation:

$$\mathbb{H}\boldsymbol{\zeta}(x, t) = \frac{\partial}{\partial t}\boldsymbol{\zeta}(x, t), \tag{5.34a}$$

$$\boldsymbol{\zeta}(x, t) := \begin{pmatrix} f(x, t) \\ \dot{f}(x, t) \end{pmatrix}, \qquad \mathbb{H} := \begin{pmatrix} \mathbb{0} & \mathbb{1} \\ c^2\nabla^2 & \mathbb{0} \end{pmatrix}. \tag{5.34b}$$

The first component of (5.34a) states that $\dot{f}(x, t) = \partial f(x, t)/\partial t$, while the second component rewrites (5.15) in terms of f and $\dot{f}$.

5.2.8. Hyperdifferential Form of the Evolution Operator

Following step by step the formal development (2.108)–(2.113) (with $M \to 1$, $k \to c^2$, $\Delta \to \nabla^2$), we can expand the time development of the elongation and velocity functions as

$$\begin{aligned} \boldsymbol{\zeta}(x, t) &= \exp\left[(t - t_0)\frac{\partial}{\partial t'}\right]\boldsymbol{\zeta}(x, t')|_{t'=t_0} \\ &= \exp[(t - t_0)\mathbb{H}]\boldsymbol{\zeta}(x, t_0) \\ &= \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cosh[c(t - t_0)\nabla]\right. \\ &\qquad \left. + \begin{pmatrix} 0 & 1 \\ c^2\nabla^2 & 0 \end{pmatrix}(c\nabla)^{-1}\sinh[c(t - t_0)\nabla]\right\}\boldsymbol{\zeta}(x, t_0) \\ &=: \mathbb{G}(t - t_0)\boldsymbol{\zeta}(x, t_0). \end{aligned} \tag{5.35}$$

(Note that only positive powers of ∇ actually appear in this equation.) This defines *Green's* (i.e., the time-evolution) *operator* for the wave equation. Note that the operator entries of the 2×2 matrix $\mathbb{G}(t - t_0)$ involve only *even* powers of ∇, so that the symmetry properties of $f(x, t)$ and $\dot{f}(x, t)$ under inversions or translations in x are not affected [Eq. (4.121)]. Hence the boundary conditions (5.17) for the string of length L are unchanged, as we should expect, under time evolution.

For $f(x, t)$, the first component, Eq. (5.35) tells us that

$$f(x, t) = \{(c\nabla)^{-1} \sinh[c(t - t_0)\nabla]\}\dot{f}(x, t_0) + \cosh[c(t - t_0)\nabla]f(x, t_0), \tag{5.36}$$

while the second component is only the time derivative of this. Comparison of the hyperdifferential form (5.36) with the corresponding integral kernel form (5.25) of the time-evolution operator implies the (weak) equivalence

$$\cosh(\tau\nabla)f(x) = \tfrac{1}{2}[f(x + \tau) + f(x - \tau)]. \tag{5.37}$$

This is obvious by now due to (4.124) and $\cosh z = (e^z + e^{-z})/2$. The second equivalence implied is

$$\nabla^{-1} \sinh(\tau\nabla)f(x) = \tfrac{1}{2}\int_{x-\tau}^{x+\tau} dx' f(x'), \tag{5.38}$$

which is the antiderivative in τ of the first.

Exercise 5.16. Verify (5.38) in more detail (a) in comparison with (5.25)–(5.27a), (b) as the antiderivative in τ of (5.37), and (c) as the antiderivative in x of $\sinh \tau\nabla$ using $\sinh z = (e^z - e^{-z})/2$. Note that due to the absence of an $n = 0$ mode in the string, ∇^{-1} exists as an operator on the space of vibrating string solutions. Compare this to the Lanczos smoothing (4.62).

Exercise 5.17. Verify the *composition* of the time-translation operators

$$\mathbb{G}(t - t_1)\mathbb{G}(t_1 - t_0) = \mathbb{G}(t - t_0) \tag{5.39}$$

(a) formally as the exponential of $\mathbb{H}$, (b) as the 2×2 matrix with operator entries involving hyperbolic functions of ∇, (c) by the matrix representatives of $\mathbb{G}$ in the Fourier basis [Eqs. (5.26)], and (d) by the integral kernels (5.27). Recall Exercise 5.10.

5.2.9. Kinetic and Potential Energy in the Vibrating String

The last aspect we want to present of the vibrating string system is that of the energy present in the motion. This bears considerable resemblance to the energy in a vibrating finite lattice (Section 2.5) and some differences as well. In deriving the relation between $f(x, t)$ and the energy, we deal again with string elements Δx and then let $\Delta x \to 0$ and integrate over x. All quantities describing observables are assumed real.

The kinetic energy of the string element is one-half the mass $\mu\Delta x$ multiplied by the square of the velocity $\dot{f}(x, t)$. The whole string therefore has kinetic energy

$$\begin{aligned} E^k(t) &:= \tfrac{1}{2}\mu \int_0^L dx|\dot{f}(x, t)|^2 \\ &= \tfrac{1}{2}\mu(\dot{\mathbf{f}}(t), \dot{\mathbf{f}}(t))_L^{\circ} \\ &= \tfrac{1}{2}\mu \sum_{n\in\mathscr{Z}^+} \omega_n{}^2|\omega_n^{-1}\cos(\omega_n t)\dot{f}_n{}^{\circ} - \sin(\omega_n t)f_n{}^{\circ}|^2. \end{aligned} \tag{5.40}$$

In the second step we have used the inner product (4.134c) and in the third the corresponding Parseval identity, the sine Fourier coefficients being given by the time derivative of (5.25). For simplicity we have set $t_0 = 0, f_n{}^{\circ} := f_n{}^{\circ}(0)$ and $\dot{f}_n{}^{\circ} := \dot{f}_n{}^{\circ}(0)$.

The potential energy of the same string element is found by multiplying the net force acting on it, $-c^2\nabla^2 f(x, t)$, times the position $\alpha f(x, t)$ integrated from $\alpha = 0$ (equilibrium) to $\alpha = 1$ (actual position),

$$\begin{aligned} E^p(t) &:= -\mu c \int_0^L dx f(x, t)\nabla^2 f(x, t) \int_0^1 \alpha\, d\alpha \\ &= -\tfrac{1}{2}\mu c^2(\mathbf{f}(t), \nabla^2\mathbf{f}(t))_L^{\circ} \\ &= \tfrac{1}{2}\mu c^2 \sum_{n\in\mathscr{Z}^+} (n\pi/L)^2|\omega_n^{-1}\sin(\omega_n t)\dot{f}_n{}^{\circ} + \cos(\omega_n t)f_n{}^{\circ}|^2, \end{aligned} \tag{5.41}$$

where we have followed steps analogous to the derivation of (5.40).

5.2.10. Total and Partial Energy Conservation

The total energy in the string can be found after some algebra as

$$E := E^k(t) + E^p(t) = (\mu c^2\pi^2/2L^2) \sum_{n\in\mathscr{Z}^+} n^2(|f_n^{\circ}|^2 + \omega_n^{-2}|\dot{f}_n^{\circ}|^2) =: \sum_{n\in\mathscr{Z}^+} E_n. \tag{5.42}$$

The end result is only a function of the initial condition coefficients, and hence E is a constant of motion. Note in particular that the partial energies corresponding to the constituent normal modes [the sum of one term in (5.40) and the same-n term in (5.41)], denoted by E_n in (5.42), are *separately* conserved as well. Thus *there is no energy exchange between the normal modes.* (All these features have their exact counterpart in the finite lattice whose energy characteristics occupies Section 2.5.)

An interesting point to notice is the factor n^2 inside the sum in (5.42). If the total energy is to be finite, $f_n{}^{\circ}$ has to decrease faster than $n^{-3/2}$ with growing n (while $\dot{f}_n{}^{\circ}$ only faster than $n^{-1/2}$). This means that if $f(x, 0)$ has a discontinuity, the total energy of the string is unbounded. This is due to the fact that the normal mode energies are proportional to n^2, turning a converging partial-wave sum into a diverging energy sum. Discontinuities in velocity are allowed, however, as they produce only trapezoid-like string shapes.

Exercise 5.18. Follow Section 2.5 in showing that the normal mode and total energies are constant without the explicit calculation undertaken in (5.40)–(5.42).

Exercise 5.19. Follow Section 2.6 in finding other constants of motion for the vibrating string.

Exercise 5.20. Pick up the idea mentioned in Exercise 2.61 of defining a sesquilinear inner product in the string elongation–velocity space (5.34):

$$E(\zeta_1, \zeta_2) := (\dot{\mathbf{f}}_1, \dot{\mathbf{f}}_2)^{\circ}_{\mathscr{L}} - c^2(\mathbf{f}_1, \nabla^2\mathbf{f}_2)^{\circ}_{\mathscr{L}}. \tag{5.43}$$

As the spectrum of ∇^2 is strictly negative, the inner product (5.43) is positive. With respect to this product, the operator in (5.34b) is self-adjoint, and the time-evolution operator $\mathbb{G}$ in (5.35) is *unitary.*

Exercise 5.21. Consider the string to be immersed in a viscous fluid so that a velocity-dependent damping term is present. The governing equation is then

$$c^{-2}\frac{\partial^2}{\partial t^2}f(x, t) + \Gamma\frac{\partial}{\partial t}f(x, t) = \nabla^2 f(x, t). \tag{5.44}$$

In finding the solutions of this equation with the boundary conditions (5.17), note that the ∇^2-eigenfunction methods developed in this section apply with the difference that the angular frequencies (5.23b) will have a constant imaginary part, damping the oscillation, and a real part that is an "effective" oscillation frequency. The tools for this analysis have been given in Section 2.1. Lower frequencies become overdamped, while higher ones remain oscillatory. They are no longer multiples of a fundamental frequency, and hence the medium becomes dispersive, i.e., signals lose their shape during propagation as long waves lag behind short ones. This provides a rough model for the propagation of electromagnetic waves in an ionized medium.

Exercise 5.22. The boundary conditions (5.17) could be done away with as in analyzing a vibrating metal ring. The ∇^2 eigenfunctions are then $\exp(in\pi x/L)$, $n \in \mathscr{Z}$; the different approaches to the string can be applied to the ring with little conceptual difference.

Exercise 5.23. Assume the boundary conditions in space are that $\partial f(x, t)/\partial x$ be zero at $x = 0$ and L. Show that the spectrum and eigenvalues of ∇^2 are the set of nonnegative integers and that the eigenfunctions are cosines. The relevant expansion is thus the cosine Fourier series (4.136). Find the Green's function. Describe disturbances in terms of traveling waves: the mirror image disturbance is now equal in sign to the "real" one.

Exercise 5.24. Let the boundary conditions be "mixed": $f(0, t) = 0$ and $\partial f(x, t)/\partial x|_{x=L} = 0$. The problem is equivalent to that of an ordinary string with disturbances which are even with respect to reflections across $x = L/2$, the "real" string extending from zero to $L/2$. The freedom one has in choosing boundary conditions in the eigenvector procedure is that the constant term in the integration by parts (5.18) vanishes.

5.3. The Infinite Lattice

The study of finite N-point coupled lattices occupied Chapter 2 and was solved by the use of finite-dimensional vector space and transform methods. Since then, we have let $N \to \infty$ and found Fourier series. In this section we shall study the vibrations of a lattice composed of an infinity of discrete points: fundamental solutions, normal modes, and traveling waves. They are all $N \to \infty$ counterparts of the finite case. An "effective" propagation velocity for disturbances will be defined.

5.3.1. Equations of Motion

By infinite *lattice* we mean a system with a countable infinity of masses coupled by harmonic oscillator two-body interactions or their electric circuit analogues (Fig. 2.5). The equations governing such systems were found in Section 2.2. In the *simple* lattice, i.e., the case when all masses M and springs k are equal, when viscous and external forces are absent and only first-neighbor interactions are taken to exist, the governing system of equations for the disturbances $f_n(t)$ of the nth mass point is (2.26), i.e.,

$$M\ddot{f}_n = k(f_{n+1} - 2f_n + f_{n-1}) =: k(\Delta \mathbf{f})_n. \tag{5.45}$$

The only difference between (5.45) and (2.26) is that here the number N of masses is unbounded and n can take any integer value ($n \in \mathscr{Z}$). We expect that the coupled set of equations (5.45) will uncouple if we consider $\{f_n\}_{n\in\mathscr{Z}}$ to be the Fourier coefficients of a function $f(x)$ and perform Fourier synthesis on (5.45). The second-difference operator Δ becomes multiplication by $-4 \sin^2 x$ [see Eqs. (4.72)], turning Eq. (5.45) into

$$M\ddot{f}(x, t) = -4k \sin^2(x/2) f(x, t). \tag{5.46}$$

This is one ordinary differential equation in t for every value of $x \in (-\pi, \pi]$. Once $f(x, t)$ is found as determined by (5.46) with the usual initial conditions, the f_n's can be found by Fourier analysis (4.17b). The important point is that the "original function" and "partial-wave coefficients" are here, respectively, $\{f_n(t)\}_{n\in\mathscr{Z}}$ and $\{f(x, t)\}_{x\in(-\pi,\pi]}$. Their roles are reversed with respect to the ones they had in the last two sections.

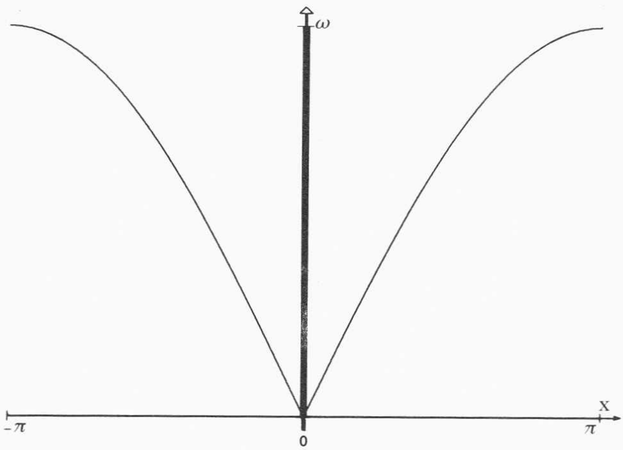

Fig. 5.8. Brillouin diagram for the oscillation angular frequencies of an infinite lattice: The allowed ω's extend from zero to $2(k/M)^{1/2}$ and are doubly degenerate for all $0 < \omega < 2(k/M)^{1/2}$.

5.3.2. Solution

If at time t_0 we state that the lattice has elongations and velocities $\{f_n(t_0), \dot{f}_n(t_0)\}_{n\in\mathscr{Z}}$, the solution to (5.46) will be determined, by the usual arguments, as

$$f(x, t) = f(x, t_0)\cos[\omega(x)(t - t_0)] + \dot{f}(x, t_0)\sin[\omega(x)(t - t_0)]/\omega(x), \quad (5.47a)$$

$$\omega(x) := 2(k/M)^{1/2}|\sin(x/2)| = \omega(2\pi - x), \quad (5.47b)$$

where $f(x, t_0)$ and $\dot{f}(x, t_0)$ are the Fourier syntheses of the initial conditions. These solutions are directly comparable with their finite-lattice counterparts (2.28), except for having a continuum of partial waves labeled by x. As x is periodic with period 2π, its range and "center" conform to Brillouin's convention. The angular frequencies $\omega(x)$ can be plotted in a Brillouin diagram (Fig. 5.8), which is the continuous counterpart of Fig. 2.10. The oscillation frequency $\omega(x)$, note, is *not* simply proportional to x, as it was for the vibrating string. This, we shall see, implies that the medium is *dispersive*: signals lose their shape as they propagate along the lattice.

5.3.3. Green's Function

The general solution to the lattice equations (5.45) can now be found as the Fourier analysis of (5.47), which is a sum of *products* of functions. As before, its structure will be that of a *convolution* between the initial conditions and the Green's function for the system and its time derivative, the latter ones being the Fourier analyses of the factors $\sin[\omega(x)(t - t_0)]/\omega(x)$ and $\cos[\omega(x)(t - t_0)]$ in (5.47), viz.,

$$f_n(t) = \sum_{m \in \mathscr{Z}} \dot{G}_{nm}(t - t_0) f_m(t_0) + \sum_{m \in \mathscr{Z}} G_{nm}(t - t_0) \dot{f}_m(t_0), \tag{5.48a}$$

$$G_{n,m}(\tau) := (2\pi)^{-1} \int_{-\pi}^{\pi} dx [\omega(x)]^{-1} \sin[\omega(x)\tau] \exp[-i(n - m)x], \tag{5.48b}$$

$$\dot{G}_{n,m}(\tau) := (2\pi)^{-1} \int_{-\pi}^{\pi} dx \cos[\omega(x)\tau] \exp[-i(n - m)x]. \tag{5.48c}$$

We have defined $G_{n,m}(\tau)$ using matrix notation, as this leads to the vector equation

$$\mathbf{f}(t) = \dot{\mathbb{G}}(t - t_0)\mathbf{f}(t_0) + \mathbb{G}(t - t_0)\dot{\mathbf{f}}(t_0), \tag{5.49}$$

in complete analogy with the expressions (2.29) for finite lattices, (5.8) for heat diffusion, and (5.35) for the vibrating string. It has in common with these systems the properties of (a) reciprocity, (b) translational invariance, and (c) invariance under inversions, as follows from noting that $G_{n,m}(\tau)$ is exclusively a function of $|n - m|$, m and n being the sites of cause and effect along the lattice. Indeed, as $[\omega(x)]^{-1} \sin \omega(x)$ is an *even*, *real* function of x, it follows that its Fourier synthesis is an even, real function of the index. As we shall see, causality, valid for a continuous medium with a definite propagation velocity, does not strictly hold here.

5.3.4. "Effective" Propagation Velocity

We now turn to the explicit calculation of the Green's functions and its time derivative, Eqs. (5.48b) and (5.48c). The integral gives rise to a transcendental function, *Bessel's* function, which is studied in Appendix B. The result can be written as

$$\dot{G}_{n,m}(\tau) = J_{2(n-m)}(2(k/M)^{1/2}\tau), \tag{5.50}$$

while $G_{n,m}(\tau)$ itself can be written as the τ-integral of (5.50) and explicitly computed by its Taylor series. In Fig. 5.9 we have plotted (5.50) for $n = 0$, integer m, and τ in a positive range. A lattice which starts from rest ($\dot{\mathbf{f}} = \mathbf{0}$) with one mass out of line with unit elongation will progress in time as shown in the figure. The disturbance propagates symmetrically on both sides of the initial elongated mass point as $J_{2k}(z) = J_{-2k}(z)$ for $k \in \mathscr{Z}$. (Fig. 5.9 should be compared with Fig. B.1, where *real* values of the index are plotted.) At time $\tau = 0$ $J_k(0) = 0$ for all k except $J_0(0) = 1$, so at $t = t_0$ (5.49) is identically satisfied. Fig. 5.9 can be seen to resemble—in a neighborhood of $\tau = 0$—the corresponding Green's function derivative for a finite lattice in

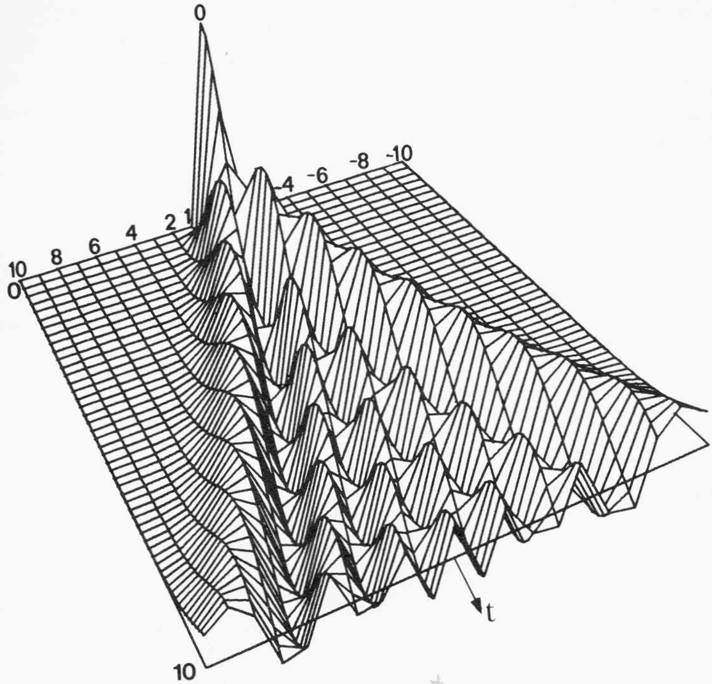

Fig. 5.9. The Green's function time derivative $\dot{G}_{n,m}(\tau)$. This represents the motion of the mass points in a lattice which starts from rest with the zeroth mass displaced. Time is given in units of $(M/4k)^{-1/2}$.

Fig. 2.8(a). The resemblance ends when the finite lattice points antipodal to the initial disturbance start to have a significant elongation, as then the motion propagates around the finite lattice but extends indefinitely along the infinite one.

It may seem paradoxical that there is actually an infinite propagation velocity for signals in the lattice. For small z, $J_{2k}(z) \simeq (z/2)^{2k}/(2k)! \neq 0$, as can be seen from the Taylor expansion in (B.7). Hence every mass point in the lattice feels the disturbance instantaneously. A "physical" lattice of masses joined by springs, of course, *does* exhibit a finite propagation velocity due to the necessarily massive springs. The nature of the Bessel function, however, allows for a working definition of a propagation velocity. As Fig. 5.9 suggests, at points far from the disturbance focus, the elongation increases slowly up to a point where it starts oscillating. This change of response happens at a time given approximately by the first zero of the Bessel function. In Chapter 6, Fig. 6.6, we have plotted the zeros of the Bessel function. For large orders it can be shown [e.g., Watson (1922, Section 15-81) and Abramowitz and Stegun (1964, Eq. 9.5.14 and the references therein)] that the first root of $J_k(x)$ has the asymptotic value $k + 1.8557571k^{1/3} + 1.033150k^{-1/3} + \cdots$, $k \to \infty$. A given mass point at n units on either side of the disturbance (for n large) crosses this equilibrium point at a time $\tau = n(k/M)^{-1/2}$, as given by (5.50) and defines thus an "*effective*" propagation velocity of $(k/M)^{1/2}$ in units of interparticle separation per unit time. [A different justification of this estimate and the treatment of dispersion is given by Weinstock (1970), Merchant and Brill (1973), and Jones (1974).]

Exercise 5.25. Consider pth-neighbor interactions through spring constants k_p along the lines of the first part of Section 2.4. Show that the only change in the formulas in this section involves the angular frequency $\omega(x)$, which instead of (5.47b) becomes

$$\omega(x) = 2\left[k_0/4M + \sum_{p=1}^{\infty} (k_p/M)\sin^2(px/2)\right]^{1/2}, \tag{5.51}$$

in complete analogy to (2.64). The Green's function now becomes rather complicated to calculate.

Exercise 5.26. Out of (5.51) we can contrive a lattice where $\omega(x) = cx$. This will lead to a *nondispersive* lattice which can be used to propagate signals without shape loss. Replacing $2\sin^2(\alpha/2)$ by $1 - \cos\alpha$, the problem is to find the appropriate k_p's. Cosine Fourier analysis of c^2x^2 provides the answer.

5.3.5. Normal Modes

The *normal modes* for the infinite lattice can be defined, as before, as the time development of the eigenfunctions of the second-difference operator $\mathbb{\Delta}$ in (5.45) or (5.46). These are the vectors of the Dirac δ-basis [recall Eq. (4.127)]. If we let the initial conditions be $\boldsymbol{\delta}_y$ first for the elongation and then for the velocity, the corresponding normal mode solutions will be given by (5.47) for $\delta(x - y)$ and Fourier analysis, namely,

$$\dot{\varphi}_n^{y+}(t) = (2\pi)^{-1/2}\cos ny\cos[\omega(y)(t - t_0)], \tag{5.52a}$$

$$\dot{\varphi}_n^{y-}(t) = (2\pi)^{-1/2}\sin ny\cos[\omega(y)(t - t_0)], \tag{5.52b}$$

$$\varphi_n^{y+}(t) = (2\pi)^{-1/2}\cos ny\sin[\omega(y)(t - t_0)]/\omega(y), \tag{5.52c}$$

$$\varphi_n^{y-}(t) = (2\pi)^{-1/2}\sin ny\sin[\omega(y)(t - t_0)]/\omega(y), \tag{5.52d}$$

where we have taken real and imaginary parts following the nomenclature of the finite lattice case (2.48). The only difference, clearly, is that the normal modes now form a continuous set labeled by $y \in (-\pi, \pi]$. Equations (5.52) represent standing waves of wavelength $\lambda_y = 2\pi/y$ in units of interparticle separation [compare with (2.53)] and oscillation angular frequency $\omega(y)$. The shortest wavelength which can be carried by the lattice happens at the edge of the first Brillouin zone, $y = \pi$, and is $\lambda_\pi = 2$ interparticle separations. In this mode, two neighboring particles oscillate in opposite directions. Beyond the first zone ($|y| > \pi$) we have no effectively shorter wavelengths for the same reason as for the finite lattice in Fig. 2.13.

Exercise 5.27. Find the normal mode solutions (5.52) proposing *separable* solutions for the equation of motion (5.45), i.e., solutions of the form $f_n(t) = \nu(n)\tau(t)$.

Exercise 5.28. The initial condition $(2\pi)^{-1/2}\exp(-imy)$ substituted into (5.48) should yield the normal mode solutions. Perform this derivation by the Bessel generating function Eq. (B4). As only *even*-order Bessel functions will appear in the sum, use $G_B(z, t) + G_B(-z, t)$. The real and imaginary parts of the result will match Eqs. (5.52).

5.3.6. Traveling Waves

The last family of vibration modes examined for finite lattices were *traveling waves* [Eqs. (2.54)]. Here, they appear as

$$\varphi_n^{y\rightleftarrows}(t) = (2\pi)^{-1/2}\sin[ny \mp \omega(y)(t - t_0)]/\omega(y), \tag{5.53a}$$

$$\dot{\varphi}_n^{y\rightleftarrows}(t) = \mp(2\pi)^{-1/2}\cos[ny \mp \omega(y)(t - t_0)], \tag{5.53b}$$

exhibiting a propagation velocity

$$v_y^{\rightleftarrows} = \pm\omega(y)/y = \pm 2(k/M)^{1/2}|\sin(y/2)|/y \tag{5.54}$$

in units of interparticle separation per unit time. (See Fig. 2.15.) Again, as for finite lattices, the main features are that longer wavelengths have higher propagation velocities (in spite of having lower oscillation frequencies; see Fig. 2.14 to dispel this apparent paradox). Shorter wavelengths propagate slower—hence signal *dispersion* occurs. The lower limit for velocities is $2(k/M)^{1/2}/\pi$ for $y = \pi$, while the upper one is $(k/M)^{1/2}$ for $y = 0$. Not surprisingly, $(k/M)^{1/2}$ was found to be the "effective" propagation velocity from the Green's function. The instantaneous response of the whole lattice to any localized disturbance stems mathematically from the expansion of a localized function in terms of "frozen" traveling waves for $t = t_0$. Each component extends over the whole lattice, and, as time is allowed to flow, the sum—initially zero everywhere except at the disturbed site—becomes nonvanishing as the different constituent waves move at their own pace.

The production of a continuous elastic medium out of a discrete, infinite lattice proceeds as in Section 3.4: we view the lattice from an increasing distance so that only the correspondingly longer partial waves are significant. All of them have, with increasing accuracy, the same propagation velocity, as we are in the "linear" region of the Brillouin diagram near $y = 0$. In the limit, we regain the characteristics of causality common to wave phenomena in one dimension.

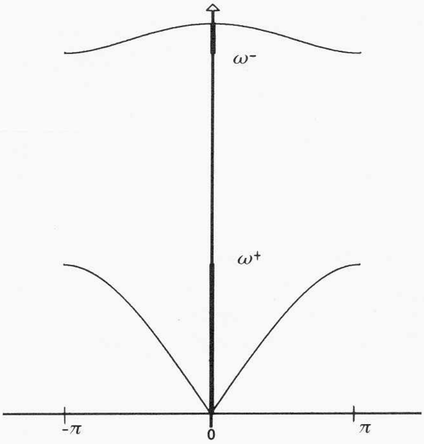

Fig. 5.10. Brillouin diagram for the oscillation angular frequencies of the molecular–diatomic infinite lattice with a spring/mass ratio of 1:2.

Exercise 5.29. Consider molecular and diatomic infinite lattices following Section 2.4 and the present description. Show that the oscillation frequency Brillouin diagram appears as in Fig. 5.10.

Exercise 5.30. Show in greater detail how Eq. (5.45), for decreasing interparticle separation, becomes the wave equation of Section 5.2. Would a molecular or diatomic lattice behave differently (exhibiting birefringence, for example)?

Exercise 5.31. Examine the *energy* in the vibration of a lattice along the lines of Section 2.5. There are no significant differences except normal modes and traveling waves have infinite energy.